A SURVEY OF KNOWN RESULTS ON 1-GENUS 1-BRIDGE KNOTS (On Heegaard Splittings and Dehn surgeries of 3-manifolds, and topics related to them)

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数理解析研究所講究録 1229 巻 2001 年 10-18

A SURVEY OF KNOWN RESULTS ON 1-GENUS 1-BRIDGE KNOTS

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We recall the definition of 1-genus 1-bridge knots. A properly imbedded arc t in a solid torus V is called *trivial* if it is boundary parallel, that is, there is a disc C imbedded in V such that $t \subset \partial C$ and $C \cap \partial V = \mathrm{cl}\,(\partial C - t)$. We call such a disc a *cancelling disc* of the trivial arc t. Let M be a closed connected orientable 3-manifold, and K a knot in M. The knot K is called a 1-*genus* 1-*bridge knot* in M if M is a union of two solid tori V_1 and V_2 glued along their boundary tori ∂V_1 and ∂V_2 and if K intersects each solid torus V_i in a trivial arc t_i for $i = 1$ and 2. The splitting $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ is called a 1-*genus* 1-*bridge splitting* of (M, K), where $H = V_1 \cap V_2 = \partial V_1 = \partial V_2$, the torus. We call also the splitting torus H a 1-*genus* 1-*bridge splitting*. We say $(1,1)$-knots and $(1,1)$-splitting for short.

1-genus 1-bridge knots are very important in light of Heegaard splittings and Dehn surgeries as shown in the theorems below.

Theorem 0.1. (T. Kobayashi [15]) *Let M be a closed orientable connected 3-manifold of genus 2. Suppose that M admits a non-trivial torus decomposition. Then either (i) M is a union os an exterior of a $(1,1)$-knot and a Seifert fibered manifold over a disc with 2-exceptional fibers, or (ii)–(v), which we omit here.*

Let $(M,K) = (V_1,t_1) \cup_H (V_2,t_2)$ be a $(1,1)$-splitting. If there are an essential simple closed curve ℓ in the torus H and cancelling discs C_i of t_i in V_i for $i=1$ and 2 such that $C_i \cap \ell = \emptyset$, then we say that the knot (M,K) has a *satellite diagram* on the $(1,1)$-splitting torus H. At this time, the knot K has a 1-bridge diagram on an annulus in H. We say that the satellite diagram is of meridional (resp. longitudinal) slope if ℓ is of meridional (resp.longitudinal) slope of V_1 or V_2.

Theorem 0.2. (K. Morimoto and M. Sakuma [19]) *Let K be a satellite knot in the 3-sphere S^3 of tunnel number one. Then K is a satellite $(1,1)$-knot such that K has a satellite diagram of non-meridional and non-longitudinal slope on the $(1,1)$-splitting torus.*

It is well-known that all the $(1,1)$-knots are of tunnel number one.

Theorem 0.3. (D. Gabai [4]) *Let V be a solid torus, and K a knot in the interior of V. Suppose that a Dehn surgery on K yields a solid torus. Then K is a 1-bridge braid, that is, isotopic to a union of an arc α on ∂V and a trivial arc in a meridian disc D of V such that all the intersection points of α and ∂D are of the same sign.*

Note that K forms a $(1,1)$-knot when we imbed the 1-bridge braid (V,K) in a standard manner in a 3-manifold of genus 1.

Theorem 0.4. (A. Thompson [27]) *Let M be a closed connected orientable 3-manifold, and $M = W_1 \cup_H W_2$ a Heegaard splitting of genus 2. Suppose that this splitting has the disjoint curve property, that is, there are an essential simple closed curve ℓ in H and essential discs D_i of the handlebody W_i such that $\ell \cap (D_1 \cap D_2) = \emptyset$ Then M is non-hyperbolic or a result of a Dehn surgery on a $(1,1)$-knot.*

These theorems show that $(1,1)$-knots are important. There are many researches on $(1,1)$-knots as below. In the following, we assume that M is not homeomorphic to $S^2 \times S^1$ for simplicity.

Let V be a solid torus, and t a trivial arc in V. We call a disc D properly imbedded in V a *t-compressing disc* if D is disjoint from t and ∂D is essential in $\partial V - \partial t$.

Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a $(1,1)$-splitting. The splitting is called *K-reducible* if there are t_i-compressing D_i in (V_i, t_i) for $i = 1$ and 2 such that $\partial D_1 = \partial D_2$ in H.

Theorem 0.5. (H. Doll [3]) *Let M be a closed connected orientable 3-manifold of genus 1, and (M, K) a $(1,1)$-knot. Then the next three conditions are equivalent.*
(1) The knot K is split, that is, the exterior of K contains an essential 2-sphere.
(2) The $(1,1)$-splitting is K-reducible.
(3) K is the trivial knot, that is, it bounds an imbedded disc in M.

He has studied more general case of g-genus n-bridge knots.

Theorem 0.6. ([9]) *Let (S^3, K) be a $(1,1)$-knot. Then K is a trivial knot if and only if the $(1,1)$-splitting is K-reducible.*

Theorem 0.7. ([9], [13], [11]) *Let (M, K) be a $(1,1)$-knot. Then K is a core knot, that is, the exterior is a solid torus if and only if for $(i, j) = (1, 2)$ or $(2, 1)$ there are a meridian disc D of V_i such that $D \cap t_i = \emptyset$ and a cancelling disc C of t_j in V_j such that ∂C intersects ∂D transversely in a single point.*

Let V be a solid torus, and t a trivial arc in V. We call a meridian disc D of V a *meridionally compressing disc* if D intersects t transversely in a single point.

Let $(M,K) = (V_1,t_1) \cup_H (V_2,t_2)$ be a $(1,1)$-splitting. The splitting is called *weakly K-reducible* if there are properly imbedded discs D_i in V_i for $i=1$ and 2 such that $\partial D_1 \cap \partial D_2 = \emptyset$ in H.

Lemma 0.8. ([10]) *Let (M,K) be a $(1,1)$-knot. Suppose that the $(1,1)$-splitting is weakly K-reducible. Then either (1) K is a core knot in a lens space, (2) K is a (maybe trivial) 2-bridge knot in S^3 or (3) K is a composite knot of a core knot and a 2-bridge knot.*

Theorem 0.9. (H. Doll [3]) *Let K be a $(1,1)$-knot. If K is a composite knot, then the $(1,1)$-splitting is weakly K-reducible.*

Theorem 0.10. (T. Kobayashi and O. Saeki [16]) *Let K be a 2-bridge knot in the 3-sphere S^3. Then any $(1,1)$-splitting of K is weakly K-reducible.*

Theorem 0.11. (K. Morimoto [18]) *Let K be a non-trivial non-core torus knot, where "torus" knot means that K can be isotoped into a Heegaard splitting torus. Then any $(1,1)$-splitting of K is cancellable, that is, there are cancelling discs C_i of t_i in V_i for $i=1$ and 2 such that $\partial C_1 \cap \partial C_2 = \partial t_1 = \partial t_2$.*

We can push K along the discs C_1 and C_2 into the splitting torus.

Theorem 0.12. ([9]) *Let (M,K) be a $(1,1)$-knot. Suppose that K is a cabled knot, that is, there is a solid torus V in M such that $K \subset \partial V$ and that any meridian disc of V intersects K in two or more points. Then either (1) the $(1,1)$-splitting is K-reducible or weakly K-reducible, (2) K is a torus knot, or (3) K has a 1-bridge diagram on an annulus A in the splitting torus H such that each bridge is an essential arc in A.*

Theorem 0.13. ([10]) *Let (M,K) be a $(1,1)$-knot. Note that M may be a lens space. If K is a satellite knot, then the $(1,1)$-split admits a satellite diagram of a non-meridional non-longitudinal slope.*

Theorem 0.14. (H. Matsuda [17]) *Let (S^3, K) be a non-trivial $(1,1)$-knot. Suppose that K bounds a Seifert surface F of genus 1. Then either (1) K is a 2-bridge knot and F is a plumbing sum of two twisted unknotted annulus or (2) F is obtained from an essential annulus A in the $(1,1)$-splitting torus H by adding a twisted band along an essential arc in $H - \partial A$.*

Theorem 0.15. (M. Hirasawa and C. Hayashi [12]) *Let $(M, K) = (V_1, t_1) \cup_H (V_2, t_2)$ be a $(1,1)$-splitting. Let F' be a closed connected orientable surface of genus 2 imbedded in M such that K is contained in F' and that F intersects the knot exterior in an incompressible and boundary incompressible surface. Then F' can be isotoped to intersect each solid torus V_i in zero or some number of ∂-parallel annuli disjoint from K and one of the surfaces of four types (a)–(b) as below :*
(a) ∂-parallel once punctured torus which contains the arc t_i,
(b) an annulus A which is parallel to an annulus A' in ∂V, contains the arc t_i, and added a non-twisted band B along an essential arc in A', so that $A \cup B$ forms a once punctured torus,
(c) a pair of pants P such that P is ∂-parallel in ∂V_i, that P contains the arc t_i, that precisely two components of ∂P is essential in ∂V, and that ∂t_i is contained in the other component of ∂P,
(d) an annulus Z which is parallel to an annulus Z' in ∂V, contains the arc t_i, and added a non-twisted band C along an inessential arc in A', so that $Q = Z \cup C$ forms a pair of pants and that the inessential component of ∂Q contains ∂t_i.

These theorems are on $(1,1)$-splittings of special $(1,1)$-knots. How about $(1,1)$-splittings of general $(1,1)$-knots?

Following theorem helps study of $(1,1)$-splittings. This is a generalization of a result by H. Rubinstein and M.Scharlemann [22].

Theorem 0.16. (T. Kobayashi and O. Saeki [16]) *Let M be a closed connected orientable 3-manifold. Let L be a link in M. Suppose that M has a 2-fold branched covering with the branched set L. Let H_i be a (g_i, n_i)-splitting of (M, L) for $i = 1$ and 2. Suppose that the splittings are not weakly L-reducible. Then after an adequate isotopy H_1 and H_2 intersect each other transversely in a non-empty collection of L-essential loops, that is, none of the loops $H_1 \cap H_2$ bounds a disc D in H_1 or H_2 such that D is disjoint from L or intersects L in a single point.*

There are some notes on the above theorem.
(1) A $(1,1)$-splitting is a special case of a (g,n)-splitting.
(2) The condition "non-empty" is very important because we can isotope H_1 and H_2 to be disjoint from each other.
(3) The projective space $\mathbb{R}P^3$ does not have a branched covering with the branched set a core knot, for example.
(4) The author expect that the above theorem holds when there is not such a branched covering.

Theorem 0.17. ([11]) *Let M be the 3-sphere S^3 or a lens space. Let K be a knot in M. Let H_1 and H_2 be $(1,1)$-splitting tori of (M, K). Suppose that H_1 and H_2 intersect each other transversely in a non-empty collection of K-essential loops. Then after an adequate isotopy either*
(1) H_1 and H_2 are isotopic to each other in (M, K),
(2) one of the splittings H_1 and H_2 is weakly K-reducible,
(3) K is a satellite knot, or
(4) H_1 and H_2 intersect each other transversely in 1 or 2 K-essential loops.

Theorem 0.18. ([11]) *In case (4) in the previous theorem, after an adequte isotopy at least one of the next four conditions (a)–(d) holds.*
(a) One of (1)–(3) in the conclusion of the previous theorem holds.
(b) (M, K) is a sum of two tangles (B, T) and (X, S) as below. (B, T)

is a trivial 2-string tangle. X is a once punctured lens space and S is a disjoint union of two arcs s_1 and s_2 properly imbedded in X such that $E_i = cl(X - N(s_i))$ is a solid torus and that s_j is parallel to the boundary ∂E_i for $(i,j) = (1,2)$ or $(2,1)$. The $(1,1)$-splitting torus H_i is obtained from ∂X by applying a tubing operation along the arc s_i for $i = 1$ and 2.
(c) One of the splittings H_1 and H_2 admits a satellite diagram of a longitudinal slope.
(d) There is a solid torus V in M as below. The exterior of the solid torus is also a solid torus. The knot K intersects V in two arcs. There are disjoint union of two discs D_1 and D_2 in ∂V as below. There are disjoint union of two balls B_1 and B_2 such that $B_i \cap V = D_i$, that $K \cap B_i$ is an arc, that K intersects the solid torus $V \cup B_i$ in a trivial arc, and that H_i is isotopic to $\partial V \cap B_i$ for $i = 1$ and 2.

In case (c), the knot K is obtained from a component L_1 of a 2-bridge link $L_1 \cup L_2$ by a Dehn surgery on the other component L_2.

The author is not satisfied with the conclusion (d).

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